Concepts in Black Holes

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This document collects the definitions and results that are more geometric and analytic in nature.

Definition (Causality)**.** A vector is *causal* if it is timelike or null. A curve is *causal* if its tangent vector is everywhere causal.

Definition (Time-orientation)**.** A spacetime is *time-orientable* if it admits a causal vector field. Another causal vector is *future-directed* if it lies in the same cone, and *past-directed* otherwise.

Definition (Extendibility)**.** A spacetime is *extendible* if it is isometric to a proper subset of another spacetime known as an *extension*.

Definition (End-point)**.** $p \in \mathcal{M}$ is a *future end-point* of a future-directed causal curve $\gamma : (a, b) \to \mathcal{M}$ if for any neighbourhood $\mathcal{O} \ni p$, $\exists t_0$ such that $\gamma(t) \in \mathcal{O}\, \forall t > t_0$. A causal curve γ is *future-inextendible* if it has no future end-points. It is *inextendible* if it is future- and past-inextendible.

Definition (Completeness)**.** A geodesic is *complete* if an affine parameter of the geodesic extends to $\pm\infty$. A spacetime is *geodesically complete* if all inextendible causal geodesics are complete.

Definition (Domain of dependence)**.** Let $(\mathcal{M}, g)$ be a time-orientable spacetime. A *partial Cauchy surface* Σ is a hypersurface on which no two points are connected by a causal curve. The *future domain of independence* of Σ is $D^+(\Sigma) := \{p \in \mathcal{M} : \text{every past-inextendible causal curve through } p \text{ intesects } \Sigma\}$. The *domain of dependence* of Σ is $D(\Sigma) := D^+(\Sigma) \cup D^-(\Sigma)$.

Definition (Global hyperbolicity)**.** A spacetime is *globally hyperbolic* if it admits a *Cauchy surface*, i.e. a partial Cauchy surface Σ such that $\mathcal{M} = D(\Sigma)$.

Definition (Maximal Cauchy development)**.** Let (Σ, h_{ab}, K_{ab}) be initial data satisfying the vacuum Hamiltonian and momentum constraints. Its *maximal Cauchy development*, unique up to diffeomorphism, is a spacetime $(\mathcal{M}, g)$ that:

1. satisfies the vacuum Einstein equation;
2. is globally hyperbolic with Cauchy surface Σ;
3. induces metric h_{ab} and extrinsic curvature K_{ab} on Σ;
4. for any other spacetime satisfying the conditions above, that spacetime is isometric to a subset of $(\mathcal{M}, g)$.

Definition (Asymptotically flat initial data)**.** An initial dataset (Σ, h_{ab}, K_{ab}) is an *asymptotically flat end* if

1. Σ is diffeomorphic to $\mathbb{R}^3 \backslash B$ where B is a closed ball centred at the origin of $\mathbb{R}^3$;

2. $h_{ij} = \delta_{ij} + \mathcal{O}(1/r)$ and $K_{ij} = \mathcal{O}\big(1/r^2\big)$ as $r \to \infty$ where $r = \sqrt{x^2}$ and x^i is the pulled back coordinates on Σ from $\mathbb{R}^3$;

3. derivatives of the expressions above hold.

An initial data set is *asymptotically flat with N ends* if it is the union of a compact subset of spacetime with N asymptotically flat ends.

Strong cosmic censorship. Let (Σ, h_{ab}, K_{ab}) be a geodesically complete and N-end asymptotically flat initial data set for the vacuum Einstein equation, then generically its maximal Cauchy development is inextendible.

Definition (Null hypersurface)**.** A *null hypersurface* is one whose normal is everywhere null.

Remarks. If n_a is normal to a null hypersurface $\mathcal{N}$, then n^a is tangent to $\mathcal{N}$.

Definition (Generators)**.** *Generators* of a null hyperface $\mathcal{N}$ are integral curves of n^a where n_a is the normal to $\mathcal{N}$.

Definition (Geodesic congruence)**.** A geodesic congruence in an open subset $\mathcal{U} \subset \mathcal{M}$ is a family of geodesics such that exactly one of them passes through each point $p \in \mathcal{U}$.

Let $B^a{}_b := \nabla_b U^a$. By re-parametrisation, one can choose the deviation vector S^a to be orthogonal to the tangent vector U^a at each point, i.e. $U \cdot S = 0$.

Choose a spacetime hypersurface Σ that intersects each geodesic in a null congruence once. Choose a null vector field N^a on Σ such that $N \cdot U = -1$ on Σ and extend it off Σ by parallel transport along the geodesic. Let $P^a_b := \delta^a_b + N^a U_b + U^a N_b$ and $\hat{B}^a{}_b := P^a_c B^c{}_d P^d_b$.

Definition (Expansion, shear, rotation)**.** The *expansion*, *shear* and *rotation* of a null geodesic congruence are

$$\theta := \hat{B}^a{}_a, \quad \hat{\sigma}_{ab} := \hat{B}_{(ab)} - \frac{1}{2}\theta P_{ab}, \quad \hat{\omega}_{ab} := \hat{B}_{[ab]},$$

i.e. $\hat{B}^a{}_b \equiv \dfrac{1}{2}\theta P_{ab} + \hat{\sigma}^a{}_b + \hat{\omega}^a{}_b$.

Definition (Trapped surface)**.** A compact, orientable, spacelike 2-surface $\mathcal{S}$ in 4-dimensional spacetime is *trapped* if both family of null geodesics orthogonal to $\mathcal{S}$ have negative expansions everywhere on $\mathcal{S}$. It is *marginally trapped* if both families have non-positive expansion everywhere on $\mathcal{S}$.

Definition (Conjugate points)**.** Points p, q on a geodesic γ are *conjugate* if there exists a non-zero *Jacobi field*, i.e. a solution of the geodesic deviation equation, along γ that vanishes at p, q.

Definition (Causal structure)**.** Let $(\mathcal{M}, g)$ be a time-orientable spacetime and $\mathcal{U} \subset \mathcal{M}$. The *chronological future* of $\mathcal{U}$ is $I^+(\mathcal{U}) := \{p \in \mathcal{M} : \text{reachable by a future-directed timelike curve starting on } \mathcal{U}\}$. The *causal future* of $\mathcal{U}$ is $J^+(\mathcal{U}) := \mathcal{U} \cup \{p \in \mathcal{M} : \text{reachable by a future-directed causal curve starting on } \mathcal{U}\}$.

Definition (Achronality)**.** $\mathcal{S} \subset \mathcal{M}$ is *achronal* is no two points on $\mathcal{S}$ are connected by a timelike curve.

Definition (Cauchy horizon)**.** The *future Cauchy horizon* of a partial Cauchy surface Σ is $H^+(\Sigma) := \overline{D^+(\Sigma)} \backslash I^-(D^+(\Sigma))$.

Definition (Asymptotic flatness)**.** A time-orientable spacetime $(\mathcal{M}, g)$ is *asymptotically flat at null infinity* if there exists a spacetime $(\bar{\mathcal{M}}, \bar{g})$ such that

1. there exists a positive function Ω on $\mathcal{M}$ such that $(\bar{\mathcal{M}}, \bar{g})$ is an extension of $(\mathcal{M}, \Omega^2 g)$;
2. within $(\bar{\mathcal{M}}$, $\mathcal{M}$ can be extended to obtain a manifold-with-boundary $\mathcal{M} \cup \partial\mathcal{M}$;
3. Ω can be extended to a function on $\bar{\mathcal{M}}$ such that $\Omega\big|_{\partial\mathcal{M}} = 0$ and $\mathrm{d}\Omega\,\big|_{\partial\mathcal{M}} \neq 0$;
4. $\partial\mathcal{M}$ is the disjoint union of two component $\mathcal{I}^+$ and $\mathcal{I}^-$ each diffeomorphic to $\mathbb{R} \times S^2$;
5. no past- (future-) directed causal curve starting on $\mathcal{M}$ intersects $\mathcal{I}^+$ $(\mathcal{I}^-)$;
6. $\mathcal{I}^\pm$ are complete.

Definition (Null infinity completeness)**.** $\mathcal{I}^\pm$ is *complete* if in a gauge where $\bar{\nabla}_a \bar{\nabla}_b \Omega = 0$ on $\mathcal{I}^\pm$ we have that the generators of $\mathcal{I}^\pm$ are complete.

Definition (Black hole, white hole)**.** Let $(\mathcal{M}, g)$ be an asymptotically flat spacetime at null infinity. The *black hole* region is $\mathcal{B} \coloneqq \mathcal{M} \backslash [\mathcal{M} \cap J^-(\mathcal{I}^+)]$ where $J^-(\mathcal{I}^+)$ is defined using the unphysical spacetime $(\bar{\mathcal{M}}, \bar{g})$. The *future event horizon* is $\mathcal{H}_+ = \dot{\mathcal{B}} \equiv \mathcal{M} \cap \dot{J}^-(\mathcal{I}^+)$. The *white hole* region is $\mathcal{W} \coloneqq \mathcal{M} \backslash [\mathcal{M} \cap J^+(\mathcal{I}^-)]$ and the *past event horizon* is $\mathcal{H}_+ = \dot{\mathcal{W}} \equiv \mathcal{M} \cap \dot{J}^+(\mathcal{I}^-)$.

Definition (Strong asymptotic predictability)**.** An asymptotically flat spacetime $(\mathcal{M}, g)$ is *strongly asymptotically predictable* if there exists an open region $\bar{\mathcal{V}} \subset \bar{\mathcal{M}}$ such that $\overline{\mathcal{M} \cap \dot{J}^-(\mathcal{I}^+)} \subset \bar{\mathcal{V}}$ and $(\bar{\mathcal{V}}, \bar{g})$ is globally hyperbolic.

The above says there is a globally hyperbolic region consisting of the non-black hole region with its horizon, so that physics is predictable on and outside the horizon.

Weak cosmic censorship. Let (Σ, h_{ab}, K_{ab}) be a geodesically complete, and asymptotically flat initial data set. Let the matter fields obey hyperbolic equations and satisfy the dominant energy condition. Then generically the maximal development of this initial data is an asymptotically flat spacetime (in particular with a complete $\mathcal{I}^+$) that is strongly asymptotically predictable.

Definition (Apparent horizon)**.** Let Σ_t be a Cauchy surface in a globally hyperbolic spacetime $(\mathcal{M}, g)$. The *trapped region* $\mathcal{T}_t$ of Σ_t is the set of points $p \in \Sigma_t$ for which there exists a trapped surface $\mathcal{S}$ with $p \in \mathcal{S} \subset \Sigma_t$. The *apparent horizon* $\mathcal{A}_t$ is the boundary of $\mathcal{T}_t$.

Definition (Killing horizon)**.** A null hypersurface $\mathcal{N}$ is a *Killing horizon* if there exists a Killing vector field ξ^a normal to $\mathcal{N}$ in its neighbourhood.

Theorem (Zeroth law of black hole mechanics)**.** *Surface gravity is constant on the future event horizon of a stationary black hole spacetime obeying the dominant energy condition.*

Theorem (First law of black hole mechanics*)**.**

$$\mathrm{d}E = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q\,.$$

Theorem (Second law of black hole mechanics)**.** *Let $(\mathcal{M}, g)$ be a strongly asymptotically predictable spacetime satisfying the field equations and obeying the null energy condition. Let $\mathcal{U} \subset \mathcal{M}$ be a globally hyperbolic region for which $\overline{J^-(\mathcal{I}^+)} \subset \mathcal{U}$ whose existence is guaranteed by strong asymptotic predictability. Let Σ_1, Σ_2 be spacelike Cauchy surfaces for $\mathcal{U}$ with $\Sigma_2 \subset J^+(\Sigma_1)$ and $\mathcal{H}_i \coloneqq \mathcal{H}^+ \cap \Sigma_i$. Then area$(\mathcal{H}_2) \geqslant$ area$(\mathcal{H}_1)$.*

Theorem (Third law of black hole mechanics*)**.** *No black holes with vanishing surface gravity exist.*